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Absolute Souslin-$\mathcal{F}$ spaces and other weak-invariants of the norm topology

Roger W. Hansell

University of Connecticut, Department of Mathematics, Storrs, CT 06269-3009, USA

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Abstract

Let $h:(A,\text{weak}) \to (B,\text{weak})$ be a homeomorphism where A and B are arbitrary subsets of (possibly different) Banach spaces. Then any property that holds for (B,norm) whenever it holds for (A,norm) is said to be a weak-invariant of the norm topology. We show that, relative to the norm topologies on A and B, the map h and its inverse are $\mathcal{F}_\sigma$-measurable and take norm discrete collections to norm σ-discretely decomposable collections. We deduce from this a number of properties that are weak-invariants of the norm topology, including such properties as being an absolute Borel space, being an absolute Souslin-$\mathcal{F}$ space, and being σ-locally of weight less than some infinite cardinal κ. The latter two properties generalize results of Namioka and Pol (1993) who showed previously that being an absolute Souslin-$\mathcal{F}$ space of weight $\leqslant \aleph_1$ and being a σ-discrete set are weak-invariants of the norm topology. Other weak-invariants such as (A,weak) being σ-fragmented by the norm are also established.

Keywords: Weak topology of Banach spaces; Souslin sets; Base-σ-discrete map; Borel measurable map

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1. Introduction

In [13] Namioka and Pol introduced the concept of a weak-invariant of the norm topology: A property (P) relative to the norm topology of a Banach space is said to be a *weak-invariant* if, whenever A and B are subsets of two Banach spaces such that (A, weak) and (B, weak) are homeomorphic, (B, norm) has property (P) whenever (A, norm) does. As noted in [13], separability and more generally the weight of the norm topology are exampls of weak-invariants. The main result in [13] is the following theorem.

E-mail address: hansell@math.uconn.edu (R.W. Hansell).

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Theorem [13]. *Let K be a compact space, let M be a metric space, and let $f : M \to (C(K), \tau_p)$ be a continuous one-to-one map. If $f(M)$ is a σ-discrete set relative to the norm topology, then M is σ-discrete.*

Here $C(K)$ denotes the Banach space of all real-valued continuous functions on K with the supremum norm, and τ_p denotes the topology of pointwise convergence on $C(K)$. It follows from the above theorem that being a σ-discrete set is a weak-invariant property of the norm topology [13, Corollary 2]. We show here that the stronger property of being σ-locally of weight less than some infinite cardinal κ, symbolized $\sigma\mathrm{LW}(<\kappa)$, is also a weak-invariant of the norm topology. It follows easily from the definitions that a Hausdorff space is σ-discrete if and only if it has $\sigma\mathrm{LW}(<\aleph_0)$. Using the fact that being σ-discrete is a weak-invariant, together with previous results of Pol and the author [5], it is shown in [13, Corollary 3] that the property of being an absolute Souslin-$\mathcal{F}$ space of weight $\leqslant \aleph_1$ is a weak-invariant property of the norm topology (the definitions are recalled below). Whether the weight restriction can be dropped was left open, although it was noted that the general result is consistent with ZFC, assuming the existence of large cardinals, based on work of Fleissner (see [13, Remark on p. 511]).

The following is the main result of the present note and will enable us to show that the property of being an absolute Souslin-$\mathcal{F}$ space of arbitrary weight is a weak-invariant of the norm topology. The definitions are given after the statement of the theorem.

Theorem 1. *Let K be a compact space, let X be a topological space, and let $f : X \to (C(K), \tau_p)$ be a continuous map. If f has a σ-relatively discrete function base for the topology τ_p, then f has a σ-relatively discrete function base of $\mathcal{F} \cap \mathcal{G}$ sets with respect to the norm topology on $C(K)$. It follows that, relative to the norm topology, f is $(\mathcal{F} \cap \mathcal{G})_\sigma$-measurable and a uniform limit of a sequence of $\mathcal{F} \cap \mathcal{G}$-simple* (*hence piecewise continuous*) *maps.*

Recall that a family $\mathcal{H}$ of subsets of a space X is *discrete* if each point of X has a neighborhood that meets at most one member of $\mathcal{H}$, and is *relatively discrete* if the condition holds relative to the subspace $\bigcup \mathcal{H}$. A family is σ-(*relatively*) *discrete* if it is countable union of (relatively) discrete families. We say that $\mathcal{B} \subset \mathcal{P}(X)$ is a *function base* for a map $f : X \to Y$ if, whenever V is open in Y, then $f^{-1}(V)$ is a union of some subcollection of $\mathcal{B}$. Maps having a σ-(relatively) discrete function base play an important role in the study of nonseparable descriptive topology (see, e.g., [4], [3, p. 309]). Note that if X has a σ-relatively discrete network for its topology (in particular, if X is metrizable), then any continuous map $f : X \to Y$ has a σ-relatively discrete (σ-discrete when X is metrizable)) function base. It follows that Theorem 1 implies the result of [13] quoted above.

By a $\mathcal{F} \cap \mathcal{G}$ set in the space X we mean a set that is the intersection of a closed set with an open set of X—equivalently, the difference of two closed (or open) sets. An $(\mathcal{F} \cap \mathcal{G})_\sigma$ set is a countable union of $\mathcal{F} \cap \mathcal{G}$ sets, and a map is $(\mathcal{F} \cap \mathcal{G})_\sigma$-*measurable* if the inverse image of open sets are $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets. Since the union of a σ-relatively discrete collection of $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets is an $(\mathcal{F} \cap \mathcal{G})_\sigma$ set [3, Theorem 6.2], it follows that any map having a σ-relatively discrete function base of $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets is $(\mathcal{F} \cap \mathcal{G})_\sigma$-measurable. Of course, if the domain of the map is such that all open sets are $\mathcal{F}_\sigma$ (in particular, if the domain is metrizable), then any $(\mathcal{F} \cap \mathcal{G})_\sigma$ measurable map is of Borel class 1 (i.e., $\mathcal{F}_\sigma$-measurable). A map $f : X \to Y$ is $\mathcal{F} \cap \mathcal{G}$-*simple* if X has a σ-relatively discrete partition $\mathcal{P}$ into $\mathcal{F} \cap \mathcal{G}$ sets such that the restriction of f to each member of $\mathcal{P}$ is contant. We show below (see Lemma 1) that any map with a σ-relatively discrete function base of $\mathcal{F} \cap \mathcal{G}$ sets and taking values in a metric space is a uniform limit of a sequence of $\mathcal{F} \cap \mathcal{G}$-simple maps. Note that if $\mathcal{H}$ is a relatively discrete collection of subsets of a space X, then each member of $\mathcal{H}$ is open in the subspace $\bigcup \mathcal{H}$, hence any map defined on X that is constant on each member of $\mathcal{H}$ is a continuous map when restricted to $\bigcup \mathcal{H}$. Thus, any $\mathcal{F} \cap \mathcal{G}$-simple map $f : X \to Y$ is *piecewise continuous* in the sense that X is a countable union of $\mathcal{F} \cap \mathcal{G}$ sets X_n such that the restriction of f to each of the subspaces X_n is continuous.

In order to state our main result on weak-invariants we need to recall one further property of maps. Following Michael [11] we call a map $f : X \to Y$ *base-σ-discrete* if, whenever $\mathcal{E}$ is a discrete collection of subsets of X, $f(\mathcal{E})$ has a σ-discrete base in the sense that there is a σ-discrete collection $\mathcal{A}$ of subsets of Y such that each of the sets $f(E)$ $(E \in \mathcal{E})$ is a union of sets from $\mathcal{A}$. Note that a bijection f between metrizable spaces is base-σ-discrete if and only if f^{-1} has a σ-discrete function base. This is a key fact in proving that the property of being an absolute Souslin-$\mathcal{F}$ space is a weak invariant. In [5] it is shown that it is exactly this property of being base-σ-discrete (called “co-σ-discrete” in [5]) that is required for Borel measurable maps to preserve absolute Souslin-$\mathcal{F}$ spaces, and a similar remark applies to one-to-one measurable maps preserving absolute (extended) Borel spaces (the definitions and precise results are recalled in Lemma 2 below). Those results together with Theorem 1 enable us to prove the following theorem on weak-invariants of the norm topology.

Theorem 2. *Let A and B be subsets of two Banach spaces such that* $(A, \textit{weak})$ *and* $(B, \textit{weak})$ *are homeomorphic. Then there is a bijection* $f : (A, \|\cdot\| \to (B, \|\cdot\|)$ *such that both* f *and* f^{-1} *are* $\mathcal{F}_\sigma$*-measurable and base-σ-discrete relative to the norm topologies. It follows that each of the following properties is a weak-invariant of the norm topology*:

(a) *absolute Borel space*;
(b) *absolute extended-Borel space*;
(c) *absolute Souslin-$\mathcal{F}$ space*;
(d) $\sigma LW(< \kappa)$, *for any infinite cardinal* κ.

We mention two other Banach space properties that are weak-invariants as a result of our work in [6]. A family $\mathcal{H}$ of disjoint subsets of a topological space X is said to be *scattered* if for any nonempty $\mathcal{E} \subset \mathcal{H}$, some $E \in \mathcal{E}$ is open relative to $\bigcup \mathcal{E}$. A subset A of a Banach space is said to be *σ-fragmented by the norm* if for every $\varepsilon > 0$ we can express A as a countable union of sets A_n so that, whenever $\emptyset \neq E \subset A_n$, there is a weak open set U such that $U \cap E \neq \emptyset$ and norm-diam$(U \cap$

$E) < \varepsilon$. This concept is due to Jayne et al. and first appeared in their preprint [8]. In [6] we studied Banach spaces whose weak topology had a σ-scattered network. In particular, it was shown that a subset A of a Banach space is σ-fragmented by the norm if and only if (A, weak) has a σ-scattered network [6, Theorem 1.12]. It follows immediately from this characterization that the property of a subset A of a Banach space being σ-fragmented by the norm is a weak-invariant of the norm topology.

In [6] we also studied the properties of a Banach space whose weak topology had a σ-relatively discrete network. Among other things, it was shown that a Banach space Z had this property if its unit sphere did, that in such Banach spaces the norm and weak Borel sets coincide, and (Z, weak) embedds as a $(\mathcal{F}\cap\mathcal{G})_{\sigma\delta}$ in $(Z^{**}, \text{weak*})$. Moreover, it was shown that a wide class of Banach spaces with "nice" equivalent norms had this property. Later, in [9], the definiton of a subset A being norm σ-fragmentable was strenthened by requiring that each point of A_n (in the notation of the above definition) be contained in a weak open set U such that norm-diam$(U \cap A_n) < \varepsilon$. In this case, A is said to have a *countable cover by sets of small local norm-diameter* ($\|\cdot\|$-SLD for short). Using arguments along the lines of the equivalence mentioned in the preceding paragraph, it can be shown that a subset A of a Banach space is $\|\cdot\|$-SLD if and only if (A, weak) has a σ-relatively discrete network. We refer the reader to the recent paper of Moltó et al. [12] for a proof of this result (due to Oncina) and for an up-to-date account of how the two network properties discussed here are important in the study of renorming properties of nonseparable Banach spaces. Summarizing, we can state the following result on weak invariants.

Theorem 3. *The properties of a subset of a Banach space being σ-fragmented by the norm or being* $\|\cdot\|$-SLD *are both weak invariants of the norm topology.*

2. Proof of Theorem 1

We begin by first proving a lemma.

Lemma 1. *Let $f : X \to Y$ be a map with a σ-relatively discrete function base.*

(a) *If f is continuous and Y is regular, then f has a σ-relatively discrete function base of $\mathcal{F}\cap\mathcal{G}$ sets.*

(b) *If f has a σ-relatively discrete function base of $\mathcal{F}\cap\mathcal{G}$ sets, then f is a pointwise limit of a sequence of $\mathcal{F}\cap\mathcal{G}$-simple maps. Furthermore, if Y is a metric space, then the convergence can be taken to be uniform.*

Proof. (a) Assume $f : X \to Y$ is continuous, Y is regular, and $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ is a function base for f with $\mathcal{H}_n$ relatively discrete for each n. Fixing n, for each $H \in \mathcal{H}_n$ let U_H be an open set in X such that $H \subset U_H$ and, whenever $H' \in \mathcal{H}_n$ and $H' \neq H$, then $H' \cap U_H \neq \emptyset$. Then it is easy to see that

$$\mathcal{H}_n^* = \left\{\overline{H} \cap U_H\colon\ H \in \mathcal{H}_n\right\}$$

is a relatively discrete collection of $\mathcal{F}\cap\mathcal{G}$ sets for each $n=1,2,\ldots$, where the bar indicates closure in the space X. Let us show that $\bigcup_{n=1}^{\infty}\mathcal{H}_n^*$ is also a function base for f. To see this let $x\in f^{-1}(V)$, for some open set $V\subset Y$. Since Y is regular, there exists an open set $W\subset Y$ such that $f(x)\in W\subset\overline{W}\subset V$. Since $\mathcal{H}$ is a function base, there exists $H\in\mathcal{H}$ such that $x\in H\subset f^{-1}(W)$. Thus, by the continuity of f, we have

$$x\in\overline{H}\cap U_H\subset f^{-1}(\overline{W})\subset f^{-1}(V)$$

as required.

To prove (b), let $\mathcal{H}$ be as in part (a) and assume all members of $\mathcal{H}$ are $\mathcal{F}\cap\mathcal{G}$ sets in X. Then, for each n, $\bigcup\mathcal{H}_n$ is an $\mathcal{F}\cap\mathcal{G}$ set in X [3, Theorem 6.2(b)], hence

$$X\setminus\bigcup\mathcal{H}_n=F_n\cup G_n,$$

where F_n is closed and G_n is open in X, and we may assume that $F_n\cap G_n=\emptyset$. Define

$$\mathcal{H}_n'=\bigl\{H_1\cap\cdots\cap H_n\colon\ H_i\in\mathcal{H}_i\cup\{F_i,G_i\},\ i=1,2,\ldots\bigr\}.$$

Then $\mathcal{H}_n'$ is a σ-relatively discrete partition of X into $\mathcal{F}\cap\mathcal{G}$ sets for each n. For each nonempty $H\in\mathcal{H}_n'$ fix a point $x_H\in H$ and define $f_n:X\to Y$ by

$$f_n(x)=f(x_H)\text{ if and only if }x\in H\quad(n=1,2,\ldots).$$

Clearly, each f_n is $\mathcal{F}\cap\mathcal{G}$-simple. To see that the sequence $\langle f_n(x)\rangle$ converges to $f(x)$ for each $x\in X$, suppose $f(x)\in V$ for some open set $V\subset Y$. By the property of a function base there exists an m and $H_m\in\mathcal{H}_m$ such that $x\in H_m\subset f^{-1}(V)$. It follows that for all $n\geqslant m$, if $x\in H\in\mathcal{H}_n'$, then $x_H\in H\subset H_m$, hence $f_n(x)=f(x_H)\in f(H_m)\subset V$ as required.

Now suppose Y is a metric space and let $B(y,\varepsilon)$ denote the open ball about y of radius ε. For a given n, we can use the function base to find a refinement $\bigcup_{m=1}^{\infty}\mathcal{E}_m$ of $\{f^{-1}(B(y,1/2n)\colon\ y\in Y\}$ where each $\mathcal{E}_m$ is a relatively discrete collection of $\mathcal{F}\cap\mathcal{G}$ sets. Using once again the fact that $X\setminus\bigcup\mathcal{E}_m=F_m\cup G_m$, where F_m and G_m are disjoint $\mathcal{F}\cap\mathcal{G}$ sets, we may assume that the sets in the above refinement are pairwise disjoint. Now define $f_n:X\to Y$ by $f_n(x)=f(x_E)$ if $x\in E\in\mathcal{E}_n$, where x_E is some fixed point in E. Again, the maps f_n are $\mathcal{F}\cap\mathcal{G}$-simple and, for any $x\in X$ both $f(x)$ and $f_n(x)$ belong to some open ball in Y of radius $1/2n$. Thus, the maps f_n converge uniformly to f. $\square$

We now turn to the proof of Theorem 1. Let K be a compact space, X any topological space, and let $f:X\to(C(K),\tau_p)$ be a continuous map having a τ_p σ-relatively discrete function base $\mathcal{H}$.

From part (a) of Lemma 1 we may assume that each member of $\mathcal{H}$ is a $\mathcal{F}\cap\mathcal{G}$ set in X. We may also assume that the range of the map f is a bounded subset of $C(K)$. To see this note that if $D_n=\{f\in C(K)\colon\ \|f\|\leqslant n\}$ $(n=1,2,\ldots)$, then D_n is τ_p closed, and so $f^{-1}(D_n)$ is closed in X. Now it is easy to verify that a map $g:X\to C(K)$ will have a norm σ-relatively discrete function base of $\mathcal{F}\cap\mathcal{G}$ sets if and only if each of the restriction maps $g|f^{-1}(D_n)$ has that property. Thus we may assume that the range of f is bounded.

Applying part (b) of Lemma 1, there exists a sequence of $\mathcal{F}\cap\mathcal{G}$-simple maps $f_n:X\to C(K)$ $(n=1,2,\ldots)$ converging pointwise to f relative to the τ_p topology. Since the range of f is (uniformly) bounded, it follows from a theorem of Grothendiek [2, Theorem 5] that the sequence $\langle f_n\rangle$ also converges pointwise to f relative to the weak topology on $C(K)$.

Now consider the sequence of maps $\langle h_n\rangle$ obtained by taking all finite rational linear combinations of $\langle f_n\rangle$. It is routine to verify that any scalar multiple and any sum of two $\mathcal{F}\cap\mathcal{G}$-simple maps taking values in a vector space (independent of any topology) are again $\mathcal{F}\cap\mathcal{G}$-simple. Thus each h_n is $\mathcal{F}\cap\mathcal{G}$-simple. Accordingly, let $\mathcal{H}_n$ be a σ-relatively discrete partition of X into $\mathcal{F}\cap\mathcal{G}$ sets such that h_n takes a constant value, say $y_H^{(n)}$, on each $H\in\mathcal{H}_n$. Given $\varepsilon>0$ let

$$M(n,\varepsilon)=\bigcup_{H\in\mathcal{H}_n} H\cap\bigl\{x\in X\colon \|f(x)-y_H^{(n)}\|\leqslant\varepsilon\bigr\}.$$

Since f is τ_p continuous, each set of the form

$$\bigl\{x\in X\colon \|f(x)-y_H^{(n)}\|\leqslant\varepsilon\bigr\}$$

is closed in X, hence each $M(n,\varepsilon)$ is an $(\mathcal{F}\cap\mathcal{G})_\sigma$ set in X as a σ-relatively discrete union of $\mathcal{F}\cap\mathcal{G}$ sets. To see that $X=\bigcup\{M(n,\varepsilon)\colon n=1,2,\ldots\}$, let x be any point in X. Since the weak and norm closures of the convex hull of $S\equiv\{f_n(x)\colon n\geqslant 1\}$ coincide and contain $f(x)$, and since the sequence $\langle h_n(x)\rangle$ is norm dense in the convex hull of S, $f(x)$ must be a norm cluster point of $\langle h_n(x)\rangle$. Thus we have $\|f(x)-h_m(x)\|\leqslant\varepsilon$ for some m. Now if $H\in\mathcal{H}_m$ contains x, then $h_m(x)=y_H^{(m)}$ and it follows that $x\in M(m,\varepsilon)$.

We now claim there exist disjoint $(\mathcal{F}\cap\mathcal{G})_\sigma$ sets $D(n,\varepsilon)$ such that

$$D(n,\varepsilon)\subset M(n,\varepsilon)\ (n=1,2,\ldots)\quad\text{and}\quad X=\bigcup_{n=1}^{\infty}D(n,\varepsilon).$$

This is possible because the collection of all finite unions of $\mathcal{F}\cap\mathcal{G}$ sets form a field (i.e., it is closed to finite unions and complementation) and so the proof of the classical reduction theorem (see [10, p. 350]) can be used verbatim here. Furthermore, each of the sets $D(n,\varepsilon)$ can be written as a countable union of disjoint $\mathcal{F}\cap\mathcal{G}$ sets, say $D(n,\varepsilon)=\bigcup_{m=1}^{\infty}D_m(n,\varepsilon)$. It follows that

$$\mathcal{H}_\varepsilon^*\equiv\bigl\{D_m(n,\varepsilon)\cap H\colon\ H\in\mathcal{H}_n \text{ and } n,m=1,2,\ldots\bigr\}$$

is a σ-relatively discrete partition of X into $\mathcal{F}\cap\mathcal{G}$ sets such that, for any $n=1,2,\ldots$ and $H\in\mathcal{H}_n$,

$$x\in D_m(n,\varepsilon)\cap H\Rightarrow\bigl\|f(x)-y_H^{(n)}\bigr\|\leqslant\varepsilon\quad(m=1,2,\ldots).$$

Thus, taking $\varepsilon=1/p$ $(p=1,2\ldots)$, we define $\mathcal{F}\cap\mathcal{G}$-simple maps $g_p\colon X\to C(K)$ by defining

$$g_p(x)=y_H^{(n)}\quad\text{if and only if}\quad x\in D_m(n,p)\cap H$$

for some $n,m=1,2,\ldots$ and $H\in\mathcal{H}_n$. It follows that $\|f(x)-g_p(x)\|\leqslant 1/p$ for every $x\in X$, hence the sequence $\langle g_p\rangle$ norm converges uniformly to f.

Finally, let us now show that the σ-discrete collection $\bigcup_{p=1}^{\infty}\mathcal{H}_p^*$ is a norm function base for f. Let $x_0\in f^{-1}(U)$ for some norm open set $U\subset C(K)$. Choose a natural number

p so that the norm-ball about $f(x_0)$ of radius $1/p$ is contained in U, and let n and m be such that $x_0 \in D_m(n, 1/2p)$. Finally, choose $H \in \mathcal{H}_n$ containing x_0. Then, for every $x \in D_m(n, 1/2p) \cap H$, we have

$$\| f(x) - f(x_0) \| \leqslant \| f(x) - y_H^{(n)} \| + \| y_H^{(n)} - f(x_0) \| \leqslant 1/p,$$

and it follows that $x_0 \in D_m(n, 1/2p) \cap H \subset f^{-1}(U)$ as required.

That completes the proof of Theorem 1. □

Remarks. The above proof uses an argument employed by Srivatsa [14, Theorem 2.1] to prove that any continuous function from a metrizable space X into a normed linear space Z with its weak topology, will be a *Baire class 1 map* relative to the norm topology on Z (i.e., a pointwise limit of a sequence of continuous maps).[1] Theorem 1 above can be viewed as a generalization of Srivatsa's theorem; in fact, Srivatsa's theorem can be deduced from it. Note first that if $\mathcal{E}$ is a relatively discrete family of $\mathcal{F} \cap \mathcal{G}$ sets in a space X in which all open sets are $\mathcal{F}_\sigma$ sets (hence, if X is metrizable), then $\mathcal{E}$ will be *σ-discretely decomposable* in X—that is, each $E \in \mathcal{E}$ can be expressed as $E = \bigcup_{n=1}^\infty E_n$ such that $\{E_n\colon E \in \mathcal{E}\}$ is discrete in X for each n. To see this, first express $\bigcup \mathcal{E}$ as an $\mathcal{F}_\sigma$, say $\bigcup \mathcal{E} = \bigcup_{n=1}^\infty F_n$, then observe that each of the families $\{F_n \cap E\colon E \in \mathcal{E}\}$ is discrete in X. Thus, if the space X in Theorem 1 is such that all open sets are $\mathcal{F}_\sigma$ sets and X has a σ-(relatively) discrete network (in particular, if X is metrizable), then any τ_p continuous map from X to $C(K)$ will have a σ-discrete function base of $\mathcal{F}_\sigma$ sets relative to the norm topology on $C(K)$ (hence is norm $\mathcal{F}_\sigma$-measurable). Now, in [4] it was shown that, if X is a collectionwise normal space and Z is any closed convex subsets of a Banach space or, alternatively, if X is metrizable and Z is any normed linear space, then $f : X \to Z$ is Baire class 1 map if and only if f has a σ-discrete function base of $\mathcal{F}_\sigma$ sets (see [4, Theorem 1.2 and Remark 1.3]). Thus, for example, if X is metrizable, then Theorem 1 implies that any τ_p continuous $f : X \to C(K)$ will have a norm σ-discrete function base of $\mathcal{F}_\sigma$ sets and hence be of norm Baire class 1.

Theorem 1 is more general in that it applies to nonmetrizable spaces as well, such as Eberlein compact spaces which have a σ-relatively discrete network.

Finally, let us observe that, if $f : X \to Y$ has a σ-discrete function base and $\mathcal{E}$ is any discrete collection in Y, then $\{f^{-1}(E)\colon E \in \mathcal{E}\}$ is σ-discretely decomposable in X. To see this let $\mathcal{H} = \bigcup_{n=1}^\infty \mathcal{H}_n$ be a function base for f where each $\mathcal{H}_n$ is discrete in X. For each $x \in f^{-1}(E)$, $E \in \mathcal{E}$, let V_x be an open set in Y containing $f(x)$ such that $V_x \cap E' = \emptyset$ for all $E' \in \mathcal{E}$ and $E' \neq E$. Now define

$$E_n = \bigcup \bigl\{ H \cap f^{-1}(E)\colon H \in y\mathcal{H}_n\colon H \subset f^{-1}(V_x) \text{ for some } x \in f^{-1}(E) \bigr\}.$$

It is routine to check that the each family $\{E_n\colon E \in \mathcal{E}\}$ is discrete in X and $f^{-1}(E) = \bigcup_{n=1}^\infty E_n$ for each $E \in \mathcal{E}$. In view of this we can deduce as an immediate corollary to Theorem 1 the principal result in [13].

[1] Namioka has informed the author that Pol has independently observed that Srivatsa's result can be used to eliminate the weight restriction in their proof that the property of being an absolute Souslin-$\mathcal{F}$ space of weight $\leqslant \aleph_1$ is a weak-invariant.

Corollary 1. *Let X be a metric space, K a compact Hausdorff space, and let $f : X \to (C(K), \tau_p)$ be a continuous one-to-one map. If $f(X)$ is σ-discrete relative to the norm topology, then X is σ-discrete.*

Since any normed linear space Z is linearly isometric to a linear subspace of $C(K)$, where K is the unit ball of the dual Z^* with the weak* topology [1, Proposition 1.4], we can also deduce as a corollary of Theorem 1 the variation on Srivatsa's theorem quoted in our remarks above. We now restate this as a corollary explicitly indicating the observation made just preceding Corollary 1, since this is one of the keys to showing that the properties listed in Theorem 2 are weak-invariants of the norm topology for Banach spaces.

Corollary 2. *Let X be a metrizable space and Z a normed linear space. Suppose $f : X \to Z$ is continuous relative to the weak topology on Z. Then, relative to the norm topology on Z, f is $\mathcal{F}_\sigma$-measurable and has a σ-discrete function base of closed sets. Consequently, if $\mathcal{E}$ is any norm discrete collection in E, then $\{f^{-1}(E) : E \in \mathcal{E}\}$ is σ-discretely decomposable in X.*

3. Proof of Theorem 2

We first recall the definitions of the spaces mentioned in Theorem 2. Recall that a subset of a topological space X is a *Souslin-$\mathcal{F}$ set* in X if it is the result of the Souslin (or $\mathcal{A}$) operation applied to the closed sets of X. By the *extended Borel sets* of a space X we mean the members of the smallest σ-algebra of subsets of X containing the closed sets and closed to the operations of complementation and σ-discrete unions. Extended Borel sets relate more naturally to the Souslin-$\mathcal{F}$ sets in nonseparable metric spaces (see [3, Section 4]). A metrizable space X is said to be *absolute Souslin-$\mathcal{F}$* (respectively, *absolute Borel* or *absolute extended Borel*) if it is a Souslin-$\mathcal{F}$ set (respectively, Borel or extended Borel set) in every metrizable space containing X topologically (equivalently, X has the given property in some completely metrizable space). Following Stone [15] for any infinite cardinal κ we say that a topological space X is *σ-locally of weight $< \kappa$* (in symbols, $\sigma\mathrm{LW}(<\kappa)$) provided X can be expressed as $\bigcup_{n=1}^{\infty} X_n$ where each X_n has the property that each point of X_n has a neighborhood of weight $< \kappa$ relative to the subspace X_n.

The following lemma describes the most general maps between metrizable spaces that preserve the character of the spaces just defined. Proofs can be found in the references cited.

Lemma 2. *Let $f : X \to Y$ be a map from the metrizable space X onto the metrizable space Y.*

(a) *If f is Souslin-$\mathcal{F}$ measurable and base σ-discrete, then Y is absolute Souslin-$\mathcal{F}$ whenever X is* [5, Theorem 7.3].
(b) *If f is one-to-one, Souslin-$\mathcal{F}$ measurable and base σ-discrete, then Y is absolute extended Borel whenever X is* [5, Theorem 7.3].

(c) *If f is one-to-one, and both f and f^{-1} are Borel measurable and base σ-discrete, then Y is absolute Borel whenever X is* [7, Theorem 11].

(d) *If f is one-to-one, and both f and f^{-1} are Souslin-$\mathcal{F}$ measurable and base σ-discrete, then Y is σ-LW$(<\kappa)$ whenever X is for any infinite cardinal κ* [15, Theorem 1′].

We now turn to the proof of Theorem 2. Let $h:(A,\ \text{weak}) \to (B,\ \text{weak})$ be a homeomorphism, where A and B are arbitrary subsets of two Banach spaces. Let f denote the composition of the three maps

$$(A, \|\cdot\|) \xrightarrow{i_A} (A,\ \text{weak}) \xrightarrow{h} (B,\ \text{weak}) \xrightarrow{i_B} (B, \|\cdot\|),$$

where we use i_X to denote the identity map on the set X. Finally, let g denote the inverse of f. Then $g:(B, \|\cdot\|) \to (A,\ \text{weak})$ is continuous, so, by Corollary 2, if $\mathcal{E}$ is any norm discrete collection of subsets of A, then $\{g^{-1}(E):\ E \in \mathcal{E}\}$ is norm σ-discretely decomposable in B. Clearly, this also shows that the map $f:(A, \|\cdot\| \to (B, \|\cdot\|)$ is base σ-discrete. Also, since $f:(A, \|\cdot\|) \to (B,\ \text{weak})$ is continuous, we may again apply Corollary 2 to conclude that $f:(A, \|\cdot\| \to (B, \|\cdot\|)$ is $\mathcal{F}_\sigma$-measurable. Thus, using the obvious symmetry, both f and f^{-1} are $\mathcal{F}_\sigma$-measurable and base σ-discrete. It now follows from Lemma 2 that the four properties listed in Theorem 2 are weak-invariants of the norm topology.

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